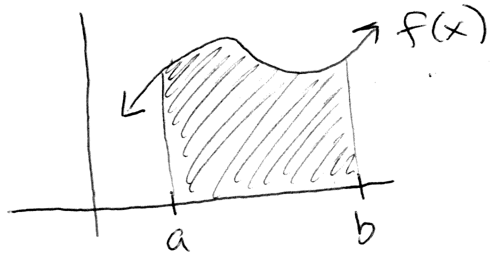
Unit 5 Applications of Integration

I. Area Under a Curve (5A)

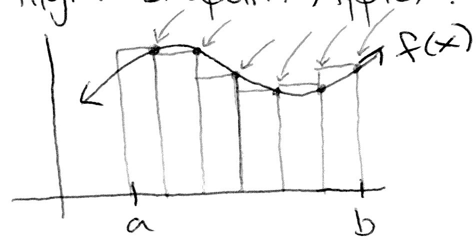


Goal: Find the area under the curve, between the endpoints.

We can't break this up perfectly, but we can make smaller rectangles to approximate the area. The more rectangles, the better the approximation.

A. 4 Types of Approximations (5B)

1. Right Endpoint Approx.

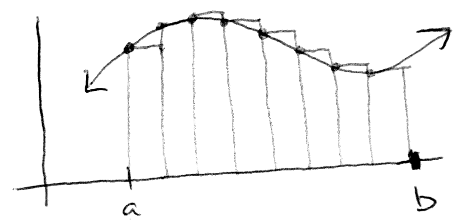


base = $\frac{b-a}{\#\text{ of rectangles}}$

height = determined by f(x)s

$A = bf(x_1) + bf(x_2) + \dots + bf(x_n)$

2. Left Endpoint Approx.

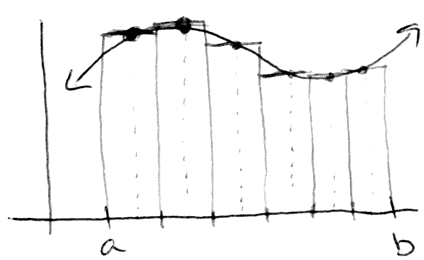


← base

← height

← area

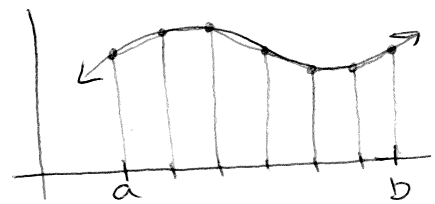
3. Midpoint Approx.

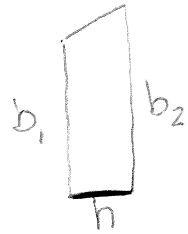


same base, height

∴ area formula...

4. Trapezoidal Approx.





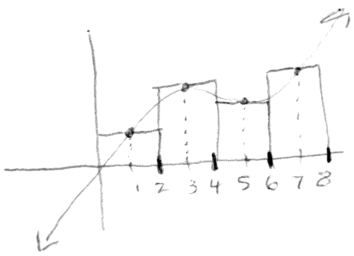
base = determined by f(x)s

height = $\frac{b-a}{\#\text{ of trapezoids}}$

$A = \frac{1}{2}h(f(x_1) + f(x_2)) + \frac{1}{2}h(f(x_2) + f(x_3))$

$+ \dots$

ex1 Use the midpoint approximation with 4 rectangles to approximate the area under the curve of $f(x) = 3x^3 + 2x$ between 0 and 8. Using 4 trapezoids?



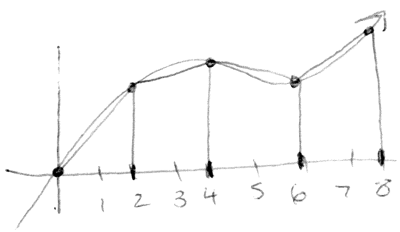
$$\text{base} = \frac{8-0}{4} = 2$$

$$\text{height} = f(x) = 3x^3 + 2x$$

$$A = 2f(1) + 2f(3) + 2f(5) + 2f(7)$$

$$2(f(1) + f(3) + f(5) + f(7)) =$$

$$2(5 + 87 + 385 + 1043) = \boxed{3{,}040}$$



$$\text{base} = f(x) = 3x^3 + 2x$$

$$\text{height} = \frac{8-0}{4} = 2$$

$$A = \cancel{\tfrac{1}{2}} \cdot \cancel{2}(f(0) + f(2)) + \cancel{\tfrac{1}{2}}\cancel{2}(f(2) + f(4)) + \cancel{\tfrac{1}{2}}\cancel{2}(f(4) + f(6))$$

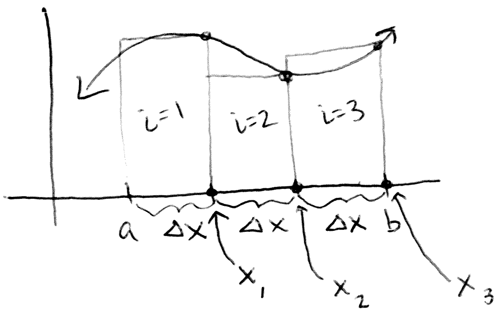
$$+ \cancel{\tfrac{1}{2}}\cancel{2}(f(6) + f(8))$$

$$f(0) + 2f(2) + 2f(4) + 2f(6) + f(8)$$

$$0 + 2 \cdot 28 + 2 \cdot 200 + 2 \cdot 660 + 1552 = \boxed{3{,}328}$$

B. Using Summation & The Limit to Find the Exact Area

To add the areas more efficiently, we can use summation notation:



Area of 1 rectangle:

$A = \Delta x \, f(x_i)$

where $\Delta x = \frac{b-a}{n}$

a = starting x-value
b = ending x-value
n = # of rectangles
i = # of the rectangle that you're "on" or calculating

$\therefore x_i = a + i\Delta x$

Summation Notation:

$$\sum_{i=1}^{n} \Delta x \, f(x_i)$$

from the 1st to the n^{th} rectangle

gives us the approximate area when you choose an n...

$$\lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \, f(x_i)$$

limit as n approaches infinity of the sum from i=1 to n of delta x times f(xᵢ).

gives the exact area!

Summation Rules

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=m}^{n} 1 = n+1-m$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

ex 1 Find the exact area under the curve of $f(x) = x+2$ from $[0,4]$.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i)$$

$$\Delta x = \frac{4-0}{n} = \frac{4}{n}$$

$$x_i = a + i\Delta x$$

$$x_i = 0 + i\left(\frac{4}{n}\right) = \frac{4i}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(\frac{4i}{n}+2\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i}{n^2} + \frac{8}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i}{n^2} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8}{n}$$

sum formula! sum formula

$$= \lim_{n\to\infty} \frac{16}{n^2} \sum_{i=1}^{n} i + \lim_{n\to\infty} \frac{8}{n} \sum_{i=1}^{n} 1 =$$

$$\lim_{n\to\infty} \frac{16}{n^2} \cdot \frac{n(n+1)}{2} + \lim_{n\to\infty} \frac{8}{n} \cdot n = \lim_{n\to\infty} \frac{8n+8}{n} + \lim_{n\to\infty} 8$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(8+\frac{8}{n}\right)}{\cancel{n}} + \lim_{n\to\infty} 8 = 8+0+8 = \boxed{16}$$

Theorem: If f is integrateable on $[a,b]$ then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i) = \int_a^b f(x)\, dx$$

area under the curve

area under the curve

II. Area Using the Definite Integral (5C)

Remember: $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i) = \int_a^b f(x)\,dx$

A. The Fundamental Theorem of Calculus

1. Part 2

> If f is continuous on $[a,b]$, then
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$
> where F is any anti-derivative of f, that is, a function such that $F' = f$

evaluate the integral at b & at a & subtract them

$\int_a^b f(x)\,dx$ represents the area under the curve (to the x-axis) from a to b, and is called the <u>definite integral</u>.

$\boxed{\text{ex1}}$ $\int_3^4 x^{-2}\,dx = \dfrac{x^{-1}}{-1} + C \Big|_3^4 = \left[\dfrac{(4)^{-1}}{-1} + C\right] - \left[\dfrac{3^{-1}}{-1} + C\right] =$

evaluated from 3 to 4

the C's will always cancel with the definite integral

$-\frac{1}{4} + \frac{1}{3} = \boxed{\frac{1}{12}}$

$\boxed{\text{ex2}}$ $\int_0^{\pi/2} \sin(x)\,dx = -\cos x \Big|_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = -0 + 1 = \boxed{1}$

B. The Definite Integral w/ U-Substitution

If you substitute out your x's for u's ...

So you must also change the bounds (a, b) to correspond to the new function.

ex 1 $\int_0^3 e^{-6x}\, dx$

$u = -6x$

$\frac{du}{dx} = -6$

$dx = \frac{du}{-6}$

bounds:

when $x = 3$

$u = -6(3) = -18$

when $x = 0$

$u = -6(0) = 0$

$\int_0^{-18} e^u \frac{du}{-6}$

$-\frac{1}{6} e^u \Big|_0^{-18} = -\frac{1}{6} e^{-18} + \frac{1}{6} e^0 = \boxed{-\frac{1}{6} e^{-18} + \frac{1}{6}}$

C. Definite Integral Rules

1. $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

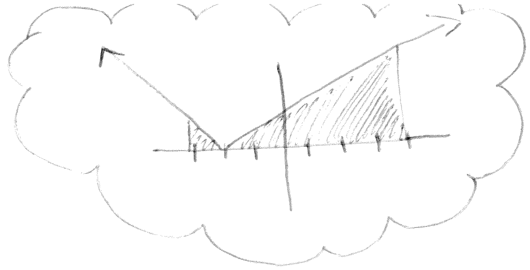
2. $\int_a^b c f(x)\, dx = c \int_a^b f(x)\, dx$

3. If $a < b$, then $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

4. If $a < b < c$, then $\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$

5. $\int_a^a f(x)\, dx = 0$

$\boxed{\text{ex 2}}$ $\int_{-3}^{4} |x+2| \, dx$



$$f(x) = \begin{cases} x+2 & x \geq -2 \\ -x-2 & x < -2 \end{cases}$$

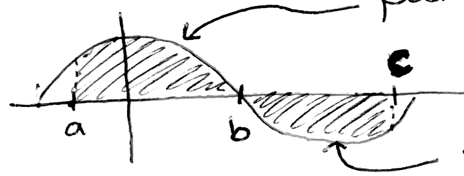
$$\int_{-3}^{-2} -x-2 \, dx + \int_{-2}^{4} x+2 \, dx$$

$$-\frac{x^2}{2} - 2x \Big|_{-3}^{-2} + \frac{x^2}{2} + 2x \Big|_{-2}^{4} = \quad \ldots \quad = \boxed{\frac{37}{2}}$$

III Area Between the Curve & the X-Axis

Sometimes the area will fall below the x-axis, what do we do?

positive heights = positive area



negative heights = negative area

but we want to <u>add</u> them together...

opposite of the negative portion to make it SUM up

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx - \int_b^c f(x)\,dx$$

$$\text{or} = \int_a^b f(x)\,dx + \int_c^b f(x)\,dx$$

switch the bounds of the negative portion.

$\boxed{\text{ex 1}}$ Find the total area bounded by the curve $y = x^3 + 3x^2 - x - 3$, the x-axis and the lines $x = -2$ and $x = 0$.

note: this is <u>NOT</u> the same as $\int_{-2}^{0} x^3 + 3x^2 - x - 3$

→ find roots:

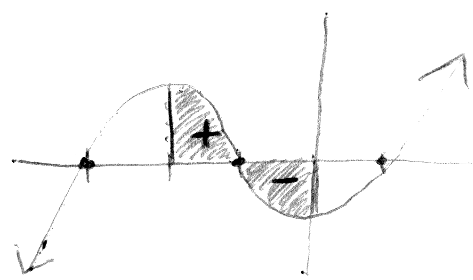
$x^3 + 3x^2 - x - 3 = 0$

$x^2(x+3) - (x+3) = 0$

$(x^2 - 1)(x+3) = 0$

$(x+1)(x-1)(x+3) = 0$

$x = -1, 1, -3$

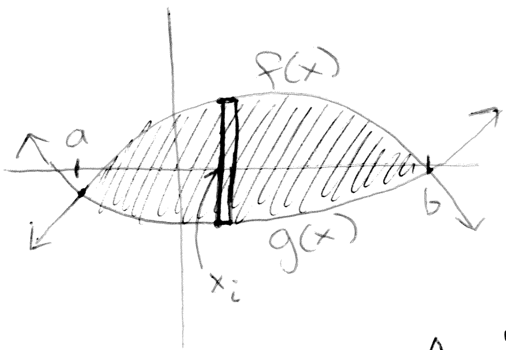


$$\int_{-2}^{-1} x^3 + 3x^2 - x - 3\,dx + \int_0^{-1} x^3 + 3x^2 - x - 3\,dx =$$

$$= \frac{x^4}{4} + x^3 - \frac{x^2}{2} - 3x \Big|_{-2}^{-1} + \frac{x^4}{4} + x^3 - \frac{x^2}{2} - 3x \Big|_{0}^{-1} = \boxed{3}$$

IV. The Area Between Two Curves (6B)

→ Suppose we no longer use the x-axis {y=0} as our boundary, but instead use a different function to contain {enclose} the area.



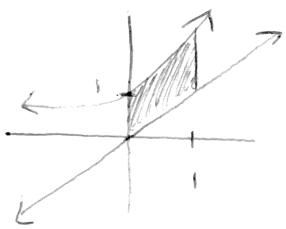
Thinking back to Riemann sums...

$$A = \Delta x \cdot h$$

$$h = f(x_i) - g(x_i)$$

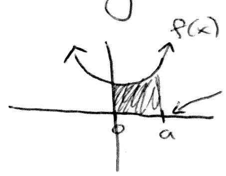
$$\therefore A = \int_a^b f(x) - g(x)\, dx$$

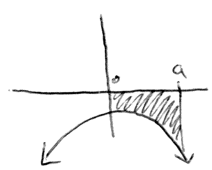
$\boxed{\text{ex1}}$ Find the area bounded by $y=e^x$, $y=x$, $x=0$ and $x=1$



$$\int_0^1 e^x - x = e^x - \frac{x^2}{2}\Big|_0^1 = \left(e - \frac{1}{2}\right) - (1-0) = \boxed{e - \frac{3}{2}}$$

connect back to using the x-axis.

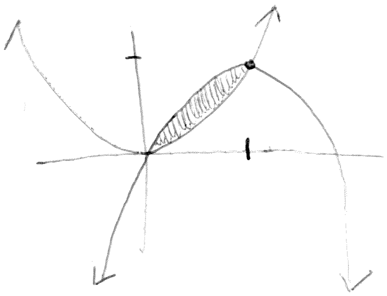




$$\int_0^a f(x) - 0\, dx \qquad \int_0^a 0 - f(x)\, dx$$

$$\int_0^a f(x)\, dx \qquad -\int_0^a f(x)\, dx$$

ex2 Find the area enclosed by the curves $y=x^2$ and $y=2x-x^2$

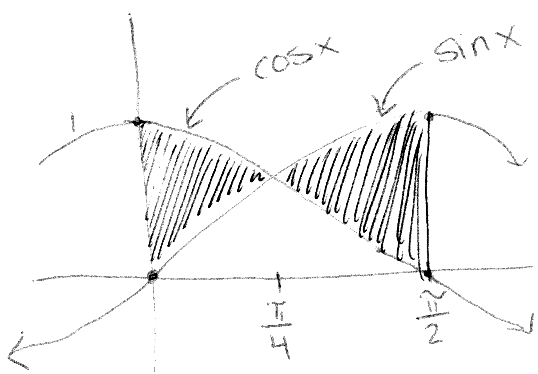


1. Find the intersections

$$x^2=2x-x^2$$
$$2x^2-2x=0$$
$$2x(x-1)=0$$
$$x=0 \quad x=1$$

$$\int_0^1 2x-x^2-x^2\,dx$$
$$\int_0^1 -2x^2+2x\,dx$$
$$-\frac{2x^3}{3}+x^2\Big|_0^1$$
$$\left(-\frac{2}{3}+1\right)-(0)=\boxed{\frac{1}{3}}$$

ex3 Find the area of the region bounded by the curves $y=\sin x$ and $y=\cos x$ and $x=0$ and $x=\frac{\pi}{2}$.



The "top" function changes...
So break the area into 2 parts.

Find the intersection:

$$\cos x=\sin x$$
$$x=\frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}} \cos x-\sin x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x-\cos x\,dx$$

$$\Big[\sin x+\cos x\Big]_0^{\frac{\pi}{4}}+\Big[-\cos x-\sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\left[\left(\sin\tfrac{\pi}{4}+\cos\tfrac{\pi}{4}\right)-\left(\sin 0+\cos 0\right)\right]+\left[\left(-\cos\tfrac{\pi}{2}-\sin\tfrac{\pi}{2}\right)-\left(-\cos\tfrac{\pi}{4}-\sin\tfrac{\pi}{4}\right)\right]$$

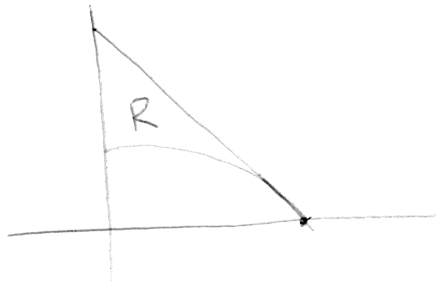
$$=\boxed{2\sqrt{2}-2}$$

Calculator: F3 → integrate f(x), x, lower bound, upper bound)

enter

AP free response question:

ex4



As shown, the region R lies in the first quadrant above the graph of $f(x) = 4 - x^2$ and below the line $y = m(x-2)$.

a. If in the first quadrant, the line lies above the graph of f, determine the range of m.

b. When the line intersects the y-axis at (0,12), what is the area of R?

c. Write an expression for A(m), the area of R in terms of m.

d. If m is decreasing at the constant rate of −2 units per second, how fast is A(m) changing at the instant the line intersects the axis at (0,12)? Is the area increasing or decreasing?

---

a. range of the slopes of the line ⟹ use the derivative of f(x) where they intersect

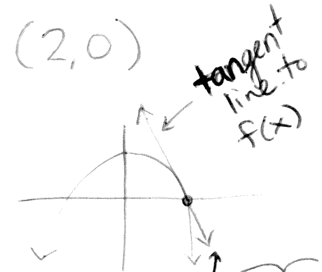
$$\frac{df}{dx} = -2x$$

f intersects the x-axis at $f(x) = (2-x)(2+x)$ (2,0)

$x = 2$ $x = -2$

the largest slope occurs here: $\frac{df}{dx} = -2(2) = -4$

$\boxed{m \leq -4}$



If the slope is any greater... the line will cut through the graph ∴ not be above it

b. $(0,12)$ to $(2,0)$ $\quad m = \frac{12-0}{0-2} = -6 \quad \therefore\ y = -6(x-2)$

$$\int_0^2 -6(x-2) - (4-x^2)\,dx = \int_0^2 -6x+12-4+x^2\,dx =$$

$$\int_0^2 x^2-6x+8\,dx = \frac{x^3}{3} - 3x^2 + 8x\Big|_0^2 = \left(\frac{8}{3} - 12 + 16\right) - (0) = \boxed{\frac{20}{3}}$$

c. $$\int_0^2 m(x-2) - (4-x^2)\,dx = \int_0^2 mx-2m-4+x^2\,dx =$$

$$\frac{mx^2}{2} - 2mx - 4x + \frac{x^3}{3}\Big|_0^2 = 2m - 4m - 8 + \frac{8}{3} = \boxed{-2m - \frac{16}{3} = A(m)}$$

d. $A(m) = -2m - \frac{16}{3}$, $\quad \frac{dm}{dt} = -2$

$$\frac{dA}{dt} = -2\frac{dm}{dt}$$

$$\frac{dA}{dt} = -2(-2) = 4$$

constant rate

The area is increasing at a rate of 4 units^2 per second

V The Fundamental Theorem of Calculus (5D)

A. Part 1: $\int_a^b f(x)\,dx = F(b) - F(a)$

B. Part 2(a):

$$\boxed{\text{If} \quad F(x) = \int_a^x f(t)\,dt \qquad \text{then} \quad F'(x) = f(x)}$$

- OR -

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

the variable must be the top bound

ex1 Compute the derivative

$$\frac{d}{dx}\int_0^x \cos(t)\,dt = \boxed{\cos x}$$

ex2 Find the derivative of

$$g(x) = \int_x^2 \sqrt{t^2+5}\,dt$$

$$g'(x) = \frac{d}{dx}\int_x^2 \sqrt{t^2+5}\,dt$$

$$\frac{d}{dx}\left[-\int_2^x \sqrt{t^2+5}\,dt\right]$$

$$= \boxed{-\sqrt{x^2+5}}$$

C. Part 2(b): $\boxed{\frac{d}{dx}\int_0^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)}$

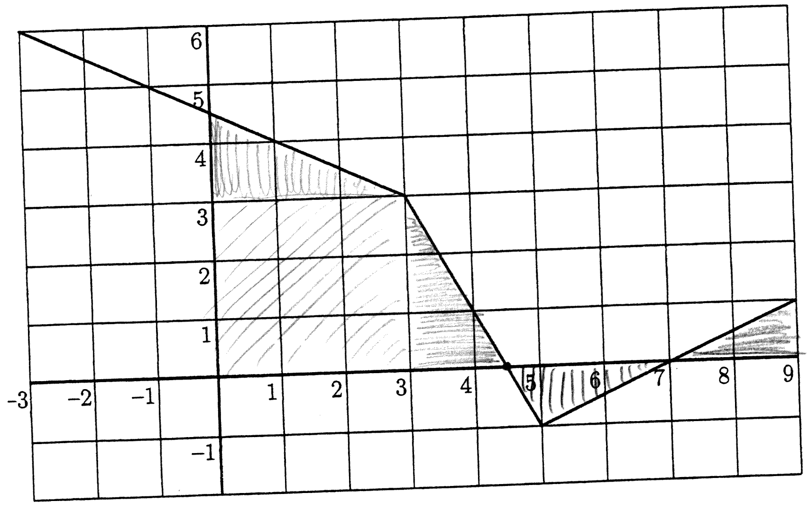
ex3 $\frac{d}{dx}\int_7^{x^2} \cos t\,dt = \boxed{2x\cos x^2}$

ex4 $\frac{d}{dx}\int_x^{x^2} t\,dt =$

$$\frac{d}{dx}\int_x^a t\,dt + \frac{d}{dx}\int_a^{x^2} t\,dt$$

$$\frac{d}{dx}\left[-\int_a^x t\,dt\right] + \frac{d}{dx}\int_a^{x^2} t\,dt = -x + 2x \cdot x^2$$

$$= \boxed{-x + 2x^3}$$



The graph of f

Let f be a function defined on the closed interval $[-3, 9]$. The graph of f, consisting of three line segments is shown above. Let $g(x) = \int_0^x f(t)\, dt$.

(a) Find $g(4.5)$, $g'(4.5)$, and $g''(4.5)$.

~~(b)~~ Find the average value of f on the interval $[-3, 5]$. Show the work that leads to your answer. (next week)

(c) Find the x-coordinate of any points of inflection of g. Justify your answer.

(d) Find the coordinates of all maximum points of g.

a. $g(4.5) = \int_0^{4.5} f(t)\, dt$ ⟹ the area under the curve from 0 to 4.5

$= \frac{1}{2}\cdot 3\cdot 1.5 + 9 + \frac{1}{2}\cdot 3 \cdot 1.5 = \boxed{13.5}$

$g'(x) = \frac{d}{dx}\int_0^x f(t)\, dt = f(x)$

$g'(4.5) = f(4.5) = \boxed{0}$

$g''(x) = f'(x)$ — Slope at 4.5

$g''(4.5) = f'(4.5) = \boxed{-2}$

c. inflection points ⟹ g'' ($f'(x)$) changes from + to − or − to +

$\boxed{x = 5}$

d. maximum ⟹ g' ($f(x)$) changes from + to − ? Check the endpoints

$x = 4.5$ $g(4.5) = 13.5$ $\boxed{(4.5, 13.5)}$

$x = 9$ $g(9) = 13.5 - \frac{1}{2}\cdot 2.5\cdot 1 + \frac{1}{2}\cdot 2\cdot 1 = 13.25$

area under the curve of f from 0 to 9

$\boxed{(9, 13.25)}$

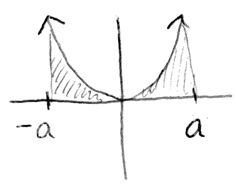
VI. Applications of the Definite Integral (6E)

A. Net Area - the area under the curve <u>with</u> + & - cancellation occuring. This will give the <u>displacement</u>. $\int_a^b f(x)\,dx$.

B. Net Change - The integral of a rate of change is called the net change. This is also referred to as the <u>accumulated</u> amount.

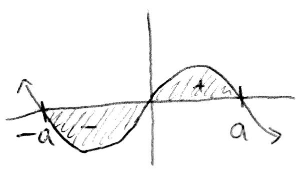
C. Total Area - the area under the curve without cancellation, add all part together. This will give the <u>total distance traveled</u>. $\int_a^b |f(x)|\,dx$.

D. Even Functions - symmetric about the y-axis



$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

E. Odd Functions - symmetric about the origin (y=x)



$$\int_{-a}^{a} f(x)\,dx = 0$$

$\boxed{\text{ex1}}$ A particle moves at a rate of $v(t) = t^2 - t - 6$ {in m/s}
Find the displacement and the distance traveled on $1 \le t \le 4$

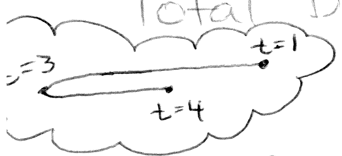
Displacement: $\int_1^4 t^2 - t - 6\,dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4 = \dots = -\frac{9}{2}$

$$\boxed{-\frac{9}{2}\text{ meters}}$$

or

$$\boxed{\frac{9}{2}\text{ meters to the left}}$$

Total Distance: $\int_1^4 |t^2-t-6|\,dt$



$= -\int_1^3 t^2-t-6\,dt + \int_3^4 t^2-t-6\,dt$

$\boxed{\frac{61}{6} \text{ meters}}$

where does it change from + to − or − to +

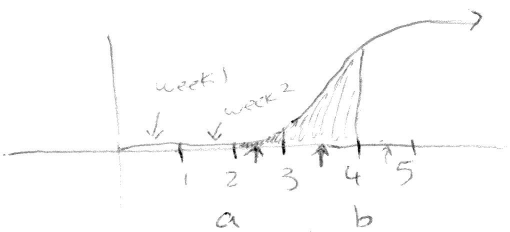
$t^2-t-6=0$

$(t-3)(t+2)=0$

$t=3 \quad \cancel{t=-2}$

int	v(t)	conclusion
(1, 3)	−	below
(3, 4)	+	above

ex2 A rumor is spreading at a rate of $R(t)=600e^{-.1t}$ people per week. Approximately how many people hear the rumor during the 3rd & 4th week?



This is a net change / accumulation problem.

$\int_2^4 600e^{-.1t}\,dt = 890.464$

$\boxed{890 \text{ people}}$

ex3 If t is measured in hours and $f'(t)$ is measured in knots, then $\int_0^2 f'(t)\,dt =$
(Note: 1 knot = 1 nautical mile/hour)

t hr
f(t) nautical mi
f'(t) nautical mi/hr

(A) $f(2)$ knots

(B) $f(2) - f(0)$ knots

(C) $f(2)$ nautical miles

(D) $f(2) - f(0)$ nautical miles

(E) $f(2) - f(0)$ knots/ hour

$\int_0^2 f'(t)\,dt$

$= f(t)\Big|_0^2$

$= f(2) - f(0)$ nautical mi

ex 4

Sand is being poured into a bin that is initially empty. During the work day, for $0 \le t < 9$ hours, the sand pours into the bin at the rate given by

$$S(t) = \frac{5000}{t^3 + 50} \text{ cubic meters per hour.}$$

After one hour, for $1 \le t < 9$, sand is removed from the bin at the rate of

$$R(t) = 23.967\sqrt{t} \text{ cubic meters per hour.}$$

(a) How much sand is put into the bin during the work day? Include units of measure.

(b) Find $S(6) - R(6)$; include units of measure. Explain what this number means in the context of the problem.

(c) Explain why the maximum amount of sand is in the bin when $S(t) = R(t)$.

(d) How much sand is in the bin at the end of the work day?

a. $\int_0^9 \frac{5000}{t^3+50}\,dt = \boxed{415.421 \text{ m}^3}$

b. $S(6) - R(6) = \frac{5000}{(6)^3+50} - 23.967\sqrt{6} = \boxed{-39.909 \text{ m}^3/\text{hr}}$

At 6 hours {t=6}, the amount of sand in the bin is decreasing at a rate of 39.909 m³/hr.

c. The maximum amount of sand in the bin is when S(t)=R(t) because this is when the rate of sand added is equal to the rate of sand removed. After that, more sand will be removed than added, so a maximum occured at that time.

- OR -

$A(t) = \int_0^t S(t)\,dt - \int_1^t R(t)\,dt$

Maximize! $A'(t) = \frac{d}{dt}\int_0^t S(t)\,dt - \frac{d}{dt}\int_1^t R(t)\,dt$

$S(t) - R(t) = 0$

$\therefore$ $S(t) = R(t)$ will find the maximum amount of sand

d. $\int_0^9 S(t)\,dt - \int_1^9 R(t)\,dt$

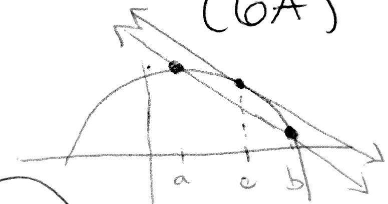
$= 415.421 - \int_1^9 23.967\sqrt{t}\,dt$

$= 415.421 - 415.428 = -.007$

$\therefore$ $\boxed{\text{the bin is empty}}$

VII The Average Value (Mean Value) of a Function (6A)

A. Prior: Mean Value Theorem



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

slope/ instantaneous value at c

average slope/average velocity on the interval [a,b]

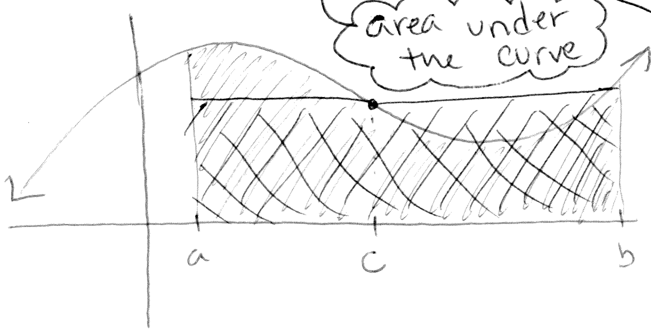
B. The Mean Value Theorem for Integrals

> If f is continuous on $[a,b]$, there exists a number c in $[a,b]$ such that:
>
> $$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$
>
> that is $\int_a^b f(x)\, dx = f(c)(b-a)$

area under the curve

height of the rectangle

base of the rectangle



C. The Average (Mean) Value Theorem

> The average value of a function $f(x)$ on the interval $[a,b]$ is
>
> $$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

don't confuse this with the average slope/velocity/rate of change

$$\frac{f(b) - f(a)}{b-a}$$

$\boxed{\text{ex1}}$ Find the average value of $f(x) = x^3 + 3x^2 + 1$ on the interval $[-2, 5]$.

$$\text{average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{5+2}\int_{-2}^{5} x^3 + 3x^2 + 1\,dx = \boxed{\frac{167}{4}}$$

$\boxed{\text{ex2}}$ A population is 40,000 and the population t months from now is estimated to be $P(t) = 40e^{.037t}$ given in thousands of people. Approximately what will be the average population over the next year?

$a = 0$
$b = 12$

$$\frac{1}{12}\int_0^{12} 40e^{.037t}\,dt = 50.354$$

$$\approx \boxed{50,354 \text{ people}}$$

VIII Volume of a Rotated Region (6C)

A. Prior Knowledge

Cube: $V = s^3$

Rectangular Prism: $V = lwh$

Cylinder: $V = \pi r^2 h$

Sphere: $V = \frac{4}{3}\pi r^3$

Cone: $V = \frac{1}{3}\pi r^2 h$

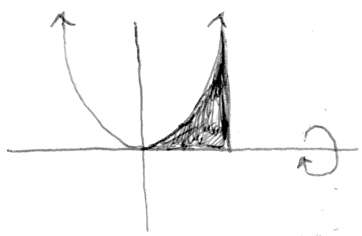
Triangular Prism: $V = \frac{1}{2} lwh$

Pyramid: $V = \frac{1}{3} lwh$

B. Volume Using the Disk Method

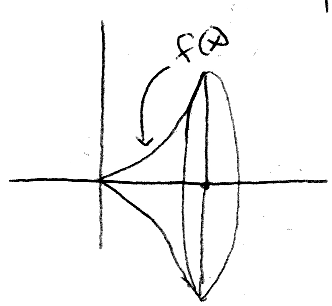
1. Rotated about a horizontal line

The area bounded by $f(x) = 3x^2$, $[0,3]$ and the x-axis rotated about the x-axis.



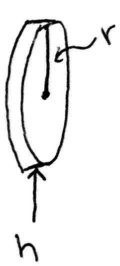
$$\text{Area} = \int_0^3 3x^2\,dx$$

As a 3-D shape:



Volume:

Slice it up into a bunch of little cylinders, aka disks



Volume of each disk: $V = \pi r^2 h$

radius = "top" − "bottom" $= f(x) - 0 = f(x)$

height $= \Delta x$

$V = \pi (f(x))^2 \Delta x$

Add all the disks:

$$\sum_{i=1}^{n} \pi (f(x))^2 \Delta x$$

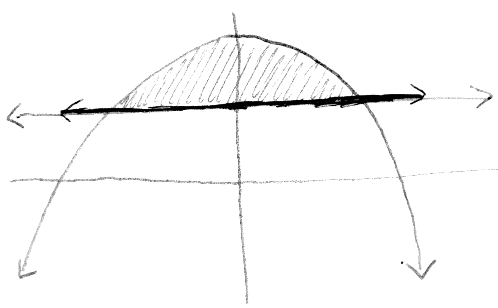
the more cylinders {n}, the more accurate the estimation

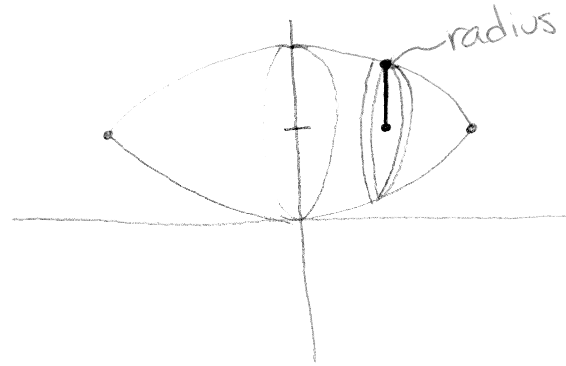
$$\lim_{n\to\infty} \sum_{i=1}^{n} \pi (f(x))^2 \Delta x = \boxed{\int_a^b \pi\, (f(x))^2\, dx}$$

radius

example: $\int_0^3 \pi\,(3x^2)^2\,dx = 9\pi \int_0^3 x^4\,dx = \boxed{\frac{2187\pi}{5}}$

$\boxed{\text{ex 1}}$ Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.





radius: "top" - "bottom" $= f(x) - g(x) = 2 - x^2 - 1 = 1 - x^2$

bounds/intersections: $2 - x^2 = 1$

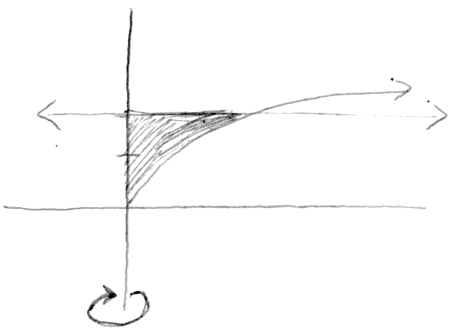
$1 = x^2$

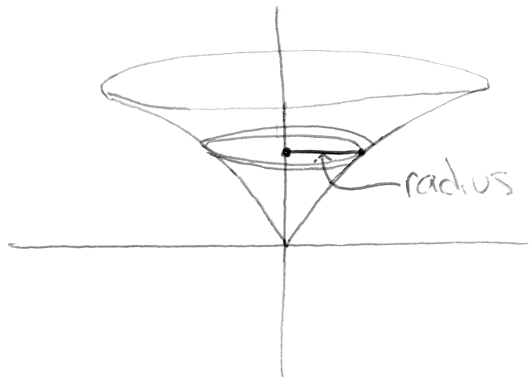
$x = \pm 1$

$$V = \int_{-1}^{1} \pi\,(1-x^2)^2\,dx = \boxed{\frac{16\pi}{5}}$$

2. Rotated about a vertical line

$\boxed{ex2}$ Find the volume of the solid formed by revolving the region bounded by $y=\sqrt{x}$, $y=2$ and $x=0$ about the y-axis.





radius: "right" − "left"

put all equations in terms of y.

$y=\sqrt{x} \Rightarrow x=y^2$

$x=0$

radius $= y^2 - 0 = y^2$

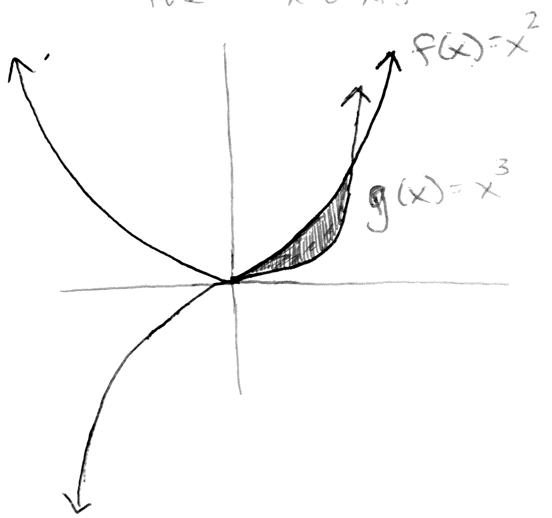
bounds: "bottom" to "top" = 0 to 2

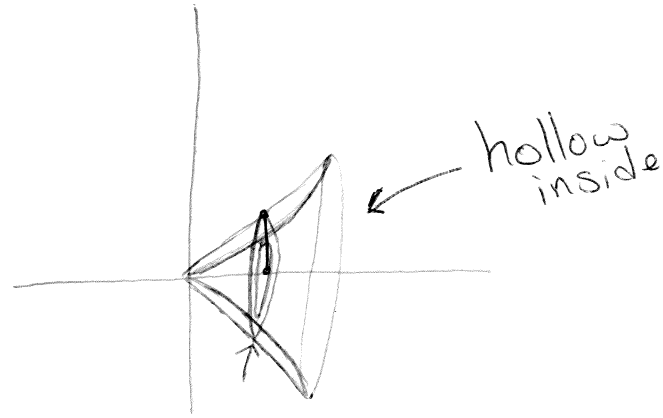
$$\int_0^2 \pi (y^2)^2 \, dy = \boxed{\frac{32\pi}{5}}$$

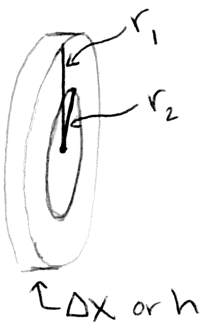
D. Washer Method

1. Rotated about a horizontal line

$\boxed{\text{ex3}}$ Find the volume of the region bounded by the curves $f(x) = x^2$ and $g(x) = x^3$ when rotated about the x-axis







Volume of each washer: $V = \pi r_1^2 h - \pi r_2^2 h$

$$\boxed{V = \int_a^b \left(\pi r_1^2 - \pi r_2^2\right) dx}$$

washer method!

example:

whole disk: $r_1 = x^2 - 0 = x^2$

hole: $r_2 = x^3 - 0 = x^3$

$$\int_0^1 \pi (x^2)^2 - \pi (x^3)^2 \, dx = \boxed{\frac{2\pi}{35}}$$

intersections/bounds

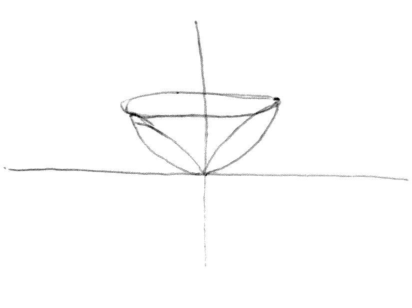
$x^3 = x^2$

$x^3 - x^2 = 0$

$x^2(x-1) = 0$

$x = 0 \quad x = 1$

2. Rotated about a vertical line



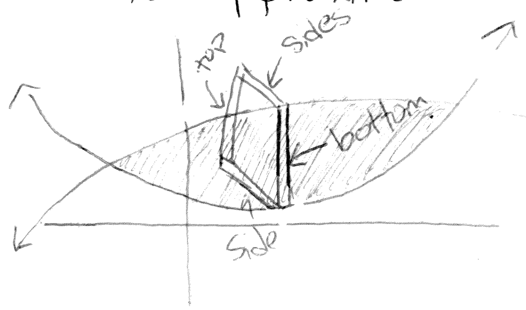
remember:

"right" - "left" to find the radii and put all equations in terms of y.

IX Volume of a Solid with a Known Cross-Section (6D)

Cross-section: a 3-D shape

we will be stacking 3-D shapes on top of a region/area aka "the base". Add the volumes of the cross-sections to approximate the volume of the entire figure.

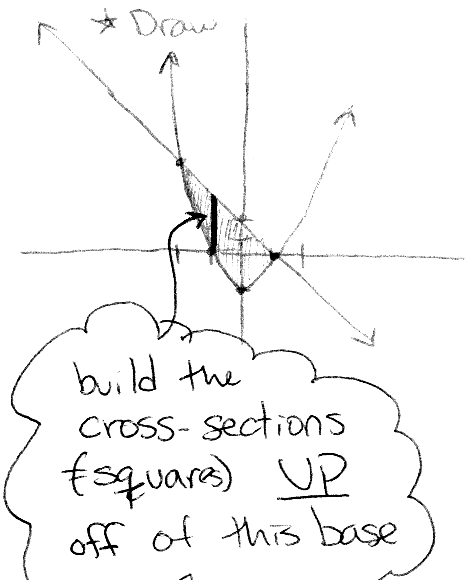


→ As the "thickness" aka Δx gets smaller, the approximation gets more accurate & the 3-D shape approaches/turns into a 2-D shape!

∴ $\int_a^b$ (the formula for area of the 2-D shape) dx

ex1 The base of a solid bounded by the curves $f(x) = 2-2x$ and $g(x) = x^2-1$. The cross-sections perpendicular to the x-axis are squares. Find the volume of the solid formed.

★ Draw



★ Formula for the area of a cross-section:

Square: $A = s^2$ ← s is given by: "top" − "bottom"

$s = (2-2x) - (x^2-1) = -x^2-2x+3$

$A = (-x^2-2x+3)^2$

★ Bounds/Intersections:

$2-2x = x^2-1$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3$ $x=1$

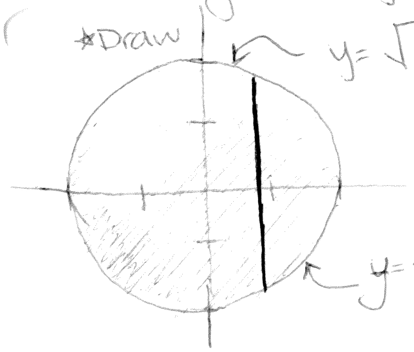
★ Volume:

$\int_{-3}^{1} (-x^2-2x+3)^2\,dx$

$= \boxed{34.133}$

ex2 Find the volume of the solid created by cross-sections that are equilateral triangles perpendicular to the x-axis using $x^2+y^2=4$ as the base.

★Draw



memorize!

★ Formula for the area equilateral triangle: $A=\frac{s^2\sqrt{3}}{4}$

$S=\sqrt{4-x^2}+\sqrt{4-x^2}=2\sqrt{4-x^2}$

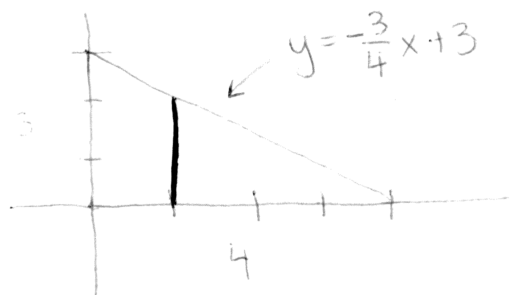
$A=\frac{(2\sqrt{4-x^2})^2\sqrt{3}}{4}$

$A=(4-x^2)\sqrt{3}$

★ Bounds:
-2 to 2

★ Volume: $V=\int_{-2}^{2}(4-x^2)\sqrt{3}\,dx=\boxed{18.475}$

ex3 Find the volume if cross-sections perpendicular to the base of a right triangle with height 3 and base 4 are semi-circles.



★ Formula for the area semi-circle $A=\frac{1}{2}\pi r^2$

$r=\frac{1}{2}(\text{"top"}-\text{"bottom"})$

$r=\frac{1}{2}\left[\left(-\frac{3}{4}x+3\right)-(0)\right]=-\frac{3}{8}x+\frac{3}{2}$

$A=\frac{1}{2}\pi\left(-\frac{3}{8}x+\frac{3}{2}\right)^2$

★ Bounds
0 to 4

★ Volume: $V=\int_0^4 \frac{1}{2}\pi\left(-\frac{3}{8}x+\frac{3}{2}\right)^2 dx$

$=\boxed{4.712}$