Unit 4 | Integrals

I. The Anti-Derivative: (4A, 4B)

Prior: Differentiation

$f(x) = x^3$

$f'(x) = 3x^2$

why? Power Rule

$f(x) = x^n$

$f'(x) = nx^{n-1}$

New: Anti-Differentiation

$f'(x) = 3x^2$

$f(x) = x^3$

why? . . .

A. Definition: A function F is called an anti-derivative of f on an interval, I, if $F'(x) = f(x)$ for all x in I.

Note: F is <u>not</u> unique, there is an infinite number of anti-derivatives.

B. Notation: Anti-Derivative / Integral

$$\int f(x)\, dx = F(x) + C$$

some constant

"the integral of . . ."   "with respect to x"

C. Rules

$$\int k \cdot f(x)\, dx = k \int f(x)\, dx$$

$$\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

constant

$\boxed{ex1}$ Evaluate $\int 3x^2\,dx$

$= 3\int x^2\,dx = \cancel{3}\cdot\frac{x^3}{\cancel{3}} + C = \boxed{x^3 + C}$

$\boxed{ex2}$ Find the anti-derivative of $f(x) = x^7 + x^4 + 3x^2$

$\int x^7 + x^4 + 3x^2\,dx = \frac{x^8}{8} + \frac{x^5}{5} + \frac{3x^3}{3} + C$

$\boxed{\frac{x^8}{8} + \frac{x^5}{5} + x^3 + C}$

$\boxed{ex3}$ Evaluate $\int (x+3)^2\,dx$

$\int x^2 + 6x + 9x^0\,dx = \boxed{\frac{x^3}{3} + \frac{6x^2}{2} + 9x + C}$

$\boxed{ex\ 4}$ Evaluate $\int 4x - \frac{2}{\sqrt[3]{x}} + \frac{5}{x^2}\,dx$

$\int 4x - 2x^{-1/3} + 5x^{-2}\,dx = \frac{4x^2}{2} - \frac{2x^{2/3}}{\frac{2}{3}} + \frac{5x^{-1}}{-1} + C$

$\boxed{2x^2 - 3x^{2/3} - 5x^{-1} + C}$

More Rules

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + c$$

$$\int e^x\,dx = e^x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \csc^2 x\,dx = -\cot x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \csc x \cot x\,dx = -\csc x + c$$

ex5 $$\int \frac{2\cot x - 3\sin^2 x}{\sin x}\,dx$$

$$\int \frac{2\cot x}{\sin x} - \frac{3\sin^2 x}{\sin x}\,dx = \int 2\cot x \csc x - 3\sin x\,dx$$

$$= \boxed{-2\csc x + 3\cos x + C}$$

II. U-Substitution (4C)

when no algebra can be done to simplify the problem into something we can take the anti derivative of... use u-substitution! ← this will help us do "backwards chain rule" type problems

ex1 $\int \tan x \, dx$

$\int \frac{\sin x}{\cos x} \, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$-\sin x \, dx = du$

$dx = \frac{du}{-\sin x}$

we do this because the entire integral needs to be in terms of u now

$\int \frac{\cancel{\sin x}}{u} \frac{du}{-\cancel{\sin x}}$

$\int -\frac{1}{u} \, du$

$-\ln u + C = \boxed{-\ln(\cos x) + C}$

this is a "chain rule type" u-sub

ex2 $\int \sqrt{1-2x} \, dx$

$u = 1-2x$

$\frac{du}{dx} = -2$

$dx = \frac{du}{-2}$

$\int \sqrt{u} \, \frac{du}{-2}$

$-\frac{1}{2} \int u^{1/2} \, du$

$-\frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}} + C\right) = -\frac{1}{3} u^{3/2} + C$

$= \boxed{-\frac{1}{3}(1-2x)^{3/2} + C}$

ex3 $\int \sin(4x)\,dx$

$u = 4x$

$\frac{du}{dx} = 4$

$\int \sin u \frac{du}{4}$

$dx = \frac{du}{4}$

$\frac{1}{4}\int \sin u\,du = \frac{1}{4}(-\cos u + c) = \boxed{-\frac{1}{4}\cos(4x) + c}$

ex 4 $\int x(2x^2+1)^6\,dx$

$u = 2x^2+1$

$\frac{du}{dx} = 4x$

$\int x \cdot u^6 \frac{du}{4\cancel{x}}$

$dx = \frac{du}{4x}$

$\frac{1}{4}\int u^6\,du = \frac{1}{4}\left(\frac{u^7}{7} + c\right) = \frac{1}{4}\cdot\frac{(2x^2+1)^7}{7} + c = \boxed{\frac{(2x^2+1)^7}{28} + c}$

ex5 $\int e^{5x}\,dx$

$u = 5x$

$\frac{du}{dx} = 5$

$\int e^u \frac{du}{5}$

$dx = \frac{du}{5}$

$= \frac{1}{5}\int e^u\,du = \frac{1}{5}(e^u + c) = \boxed{\frac{1}{5}e^{5x} + c}$

More Integral Rules

$\int a^x\,dx = \frac{a^x}{\ln x} + c$

$\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$

$\int \sec x\,dx = \ln|\sec x + \tan x| + c$

$\int \cot x\,dx = \ln|\sin x| + c$

$\int \csc x\,dx = \ln|\csc x - \cot x| + c$

all proven using u-substitution but memorize for speed!

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$$

$$\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \csc^{-1}x + c$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + c$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$$

$$\int \frac{-1}{1+x^2}\,dx = \cot^{-1}x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$\boxed{\text{ex6}}$ $\displaystyle\int 4^{-2x+3}\,dx$

$u = -2x+3$

$\frac{du}{dx} = -2$

$dx = \frac{du}{-2}$

$$\int 4^u \frac{du}{-2}$$

$$-\frac{1}{2}\int 4^u\,du = -\frac{1}{2}\left[\frac{4^u}{\ln u} + c\right] = \boxed{\frac{-4^{-2x+3}}{2\ln(-2x+3)} + c}$$

$\boxed{\text{ex7}}$ $\displaystyle\int \frac{x}{1+x^4}\,dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

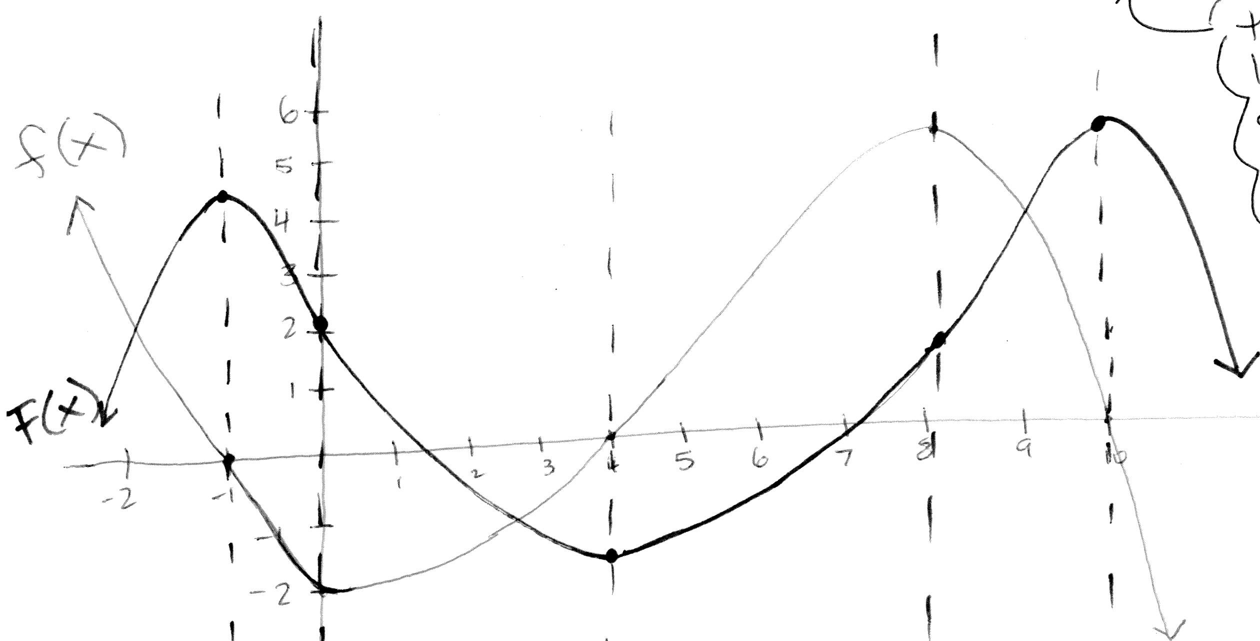
$$\int \frac{\cancel{x}}{1+u^2}\,\frac{du}{2\cancel{x}}$$

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + c = \boxed{\frac{1}{2}\tan^{-1}x^2 + c}$$

III. The Connection Between Anti-Differentiation and Curve Sketching (4A)

$\boxed{\text{ex 1}}$ Given $f(x)$, sketch the anti-derivative $F(x)$.

this is exactly like if we're given $f'(x)$ and want to sketch $f(x)$



make lines at CPs and IPs

$f(x)$	+	0	−		−	0	+	+	+	0	−
$f'(x)$	−		−	0	+		+	0	−		−
conclusion about $F(x)$	inc CD		dec CD		dec CU		inc CU		inc CD		dec CD
		max		IP		min		IP		max	

F(x) can start from any y-value because of the +C

IV. Solving Differential Equations (4D)

A. Vocabulary

Differential Equation - an equation that contains an unknown function and one or more of its derivatives.

Order - the heighest derivative in the equation

ex. $\frac{dy}{dx} = \frac{x}{y}$ is a first order differential equation

$\frac{d^2x}{dt^2} = -3x$ is a second order differential equation

$y' = 2x^2+1$ is a first order differential equation

The solution to a differential equation is a function that satisfies the differential equation when the function and its derivatives are substituted into the equation.

makes the equation true

ex. $y' = x^3$
$y = \frac{x^4}{4} + c$

because

$y' = x^3$ substitute

$\left(\frac{x^4}{4}+c\right)' = x^3$

$x^3 = x^3$ ✓

Separable equation - a first order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y.

ex. $\frac{dy}{dx} = 3x^2y$

$y' = y(3x^2+1)$

not an ex. $\frac{dy}{dx} = x^2+2y$

B. Solving Separable Equations

$\boxed{\text{ex1}}$ $\dfrac{dy}{dx} = \dfrac{2x^2}{3y^3}$

$3y^3\,dy = 2x^2\,dx$ separate

$\int 3y^3\,dy = \int 2x^2\,dx$ integrate

$\dfrac{3y^4}{4} + C = \dfrac{2x^3}{3} + C$ solve for y

$y^4 = \dfrac{8}{9}x^3 + C$

$y = \sqrt[4]{\dfrac{8}{9}x^3 + C}$

this function will satisfy $\dfrac{dy}{dx} = \dfrac{2x^2}{3y^3}$

$\boxed{\text{ex2}}$ Find all solutions to

$\dfrac{dy}{dx} = y$

$\dfrac{1}{y}\,dy = dx$

$\int \dfrac{1}{y}\,dy = \int dx$

$\ln|y| + c = x + C$

$\ln|y| = x + C$

$e^{x+C} = y$

$e^x \cdot e^C = y$

re-write e^c as C

$\boxed{Ce^x = y}$

$\boxed{\text{ex3}}$ Solve $\dfrac{dy}{dx} = y^2\sqrt{2x-3}$

$\dfrac{1}{y^2}\,dy = (2x-3)^{1/2}\,dx$

$\int y^{-2}\,dy = \int (2x-3)^{1/2}\,dx$

we need u-sub!

$-y^{-1} + C =$

$u = 2x-3$

$\dfrac{du}{dx} = 2$

$dx = \dfrac{du}{2}$

$\dfrac{1}{2}\int u^{1/2}\,du$

$\dfrac{1}{2}\left(\dfrac{2}{3}u^{3/2} + c\right)$

$-y^{-1} = \dfrac{1}{3}(2x-3)^{3/2} + C$

$$\boxed{y = \dfrac{-1}{\frac{1}{3}(2x-3)^{3/2} + C}}$$

C. Initial Conditions

In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an initial condition.

ex1 Solve $\frac{dy}{dx} = y^2(x^2+6)$ when $x=1$ and $y=6$.

we will use these to find C

$$y^{-2}dy = (x^2+6)dx$$

$$\int y^{-2}dy = \int x^2+6\,dx$$

$$-y^{-1}+C = \frac{x^3}{3}+6x+C$$

$$-y^{-1} = \frac{1}{3}x^3+6x+C$$

$$y = \frac{-1}{\frac{1}{3}x^3+6x+C}$$

use the initial condition:

$$6 = \frac{-1}{\frac{1}{3}(1)^3+6(1)+C}$$

$$6 = \frac{-1}{\frac{19}{3}+C}$$

$$38+C = -1$$

$$C = -39$$

$$\boxed{y = \frac{-1}{\frac{1}{3}x^3+6x-39}}$$

V. Applications of Differential Equations (4E)

A. Physics Applications

ex1 A particle moves in a straight line and has an acceleration of $a(t) = 6t+4$. Its inital velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find the position functi

initial position

$$a(t) = 6t+4$$

$$v(t) = \frac{6t^2}{2}+4t+C$$

$$v(0) = \frac{6(0)^2}{2}+4(0)+C = -6$$

$$C = -6$$

$$v(t) = 3t^2+4t-6$$

$$v(t) = 3t^2+4t-6$$

$$s(t) = \frac{3t^3}{3}+\frac{4t^2}{2}-6t+C$$

$$s(0) = (0)^3+2(0)^2-6(0)+C = 9$$

$$C = 9$$

$$\boxed{s(t) = t^3+2t^2-6t+9}$$

VI Slope Fields (4F)

A. A differential equation gives the slope at any given point (x, y) ; AKA it gives the slope of the tangent line to the graph at any given point.

A slope field is a graphical solution / representation to a differential equation. At each point (x, y) the slope field consists of tiny tangent lines where the slope is determined by the differential equation.

$\boxed{\text{ex 1}}$ Draw the slope field for $\frac{dy}{dx} = 2x + y$

1. Choose (x, y) s
   (The AP test will choose for you...)
2. Plug into $\frac{dy}{dx}$ to find the slopes
3. Plot tiny tangents.

x	y	$\frac{dy}{dx} = 2x + y$
0	0	0
0	1	1
0	2	2
1	0	2
1	1	3
1	2	4
2	0	4
2	1	5
2	2	6
-1	0	-2
-1	1	-1
0	-1	-1
0	-2	-2
⋮	⋮	⋮

If you're given an initial condition of $f(-1)=0$, you can plot the solutio

plot the point ; go with the flow!